

Ricci identities of the Liouville d-vector fields $z^{(1)\alpha}$ and $z^{(2)\alpha}$

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Abstract

It is the purpose of the present paper to outline an introduction in theory of embeddings in the manifold $Osc^2 M$. First, we recall the notion of 2-osculator bundle $([1],[2])$. The second section is dedicated to the notion of submanifold in the total space of the 2-osculator bundle, the manifold $Osc^2 M$. A moving frame is constructed. The induced N-linear connections and the relative covariant derivatives are discussed in third and fourth sections. The Ricci identities of the Liouville d-vector fields are present in the last section.

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Introduction

Generally, the geometries of higher order defined as the study of the category of bundles of jet $(J_0^k M, \pi^k, M)$ is based on a direct approach of the properties of objects and morphisms in this category, without local coordinates.

But, many mathematical models from Lagrangian Mechanics, Theoretical Physics and Variational Calculus used multivariate Lagrangians of higher order acceleration.

From here one can see the reason of construction of the geometry of the total space of the bundle of higher accelerations (or the osculator bundle of higher order) in local coordinates.

As far we know the theory of Finsler or Lagrange submanifolds is far from being settled. In [10] and [11] R. Miron and M. Anastasiei give the theory of subspaces in the Lagrange or generalized Lagrange spaces. Also, in [6] R. Miron presented the theory of subspaces in higher order Lagrange spaces.

This article is draw upon the original construction of the higher order geometry given by R. Miron and Gh. Atanasiu ([1],[6],[7],[8],[9]).

In [6], R. Miron construct the theory of the subspaces in higher order Lagrange spaces using the canonical metrical N-connection of the space $L^{(2)n}$

having **three** coefficients $\left(F_{jk}^i, C_{1jk}^i, C_{2jk}^i\right)$. In our work, we take the canonical metrical N-linear connection of the manifold $Osc^2 M$ having **nine** coefficients $\left(L_{(i0)bc}^a, C_{(i1)bc}^a, C_{(i2)bc}^a\right), (i = 0, 1, 2)$.

If \tilde{M} is an immersed manifold in the manifold M , a nonlinear connection N on the manifold $Osc^2 M$ induce a nonlinear connection \tilde{N} on the submanifold $Osc^2 \tilde{M}$. We take the canonical metrical N-linear connection D on the manifold $Osc^2 M$ and we obtain the induced tangent and normal connections. Also, we introduce the relative covariant derivatives in the algebra of d-tensor fields ([12]). Next, we get the Ricci identities for the Liouville d-vector fields $z^{(1)\alpha}$ and $z^{(2)\alpha}$ (Theorem 5.2). The same problem was solved by prof. Atanasiu Gh. in [1] for the Liouville d-vector fields $z^{(1)a}$ and $z^{(2)a}$ on the manifold $Osc^2 M$.

1 The 2-osculator bundle $(Osc^2 M, \pi^2, M)$

Let M be a real differentiable manifold of dimension n , whose coordinates are $(x_a)_{a=\overline{1,n}}$. Note that, throughout this paper the indices a, b, \dots run over set $\{1, 2, \dots, n\}$. The Einstein convention of summarizing is adopted all over this work.

Let us consider two curves $\rho, \sigma : I \rightarrow M$, having images in a domain of local chart $U \subset M$. We say that ρ and σ have a "contact of order 2" in a point $x_0 \in U$ if $\rho(0) = \sigma(0) = x_0, (0 \in I)$, and for any function $f \in \mathcal{F}(U)$

$$\frac{d^\beta}{dt^\beta} (f \circ \rho)(t) |_{t=0} = \frac{d^\beta}{dt^\beta} (f \circ \sigma)(t) |_{t=0}, (\beta = 1, 2). \quad (1.1)$$

The relation "contact of order 2" is an equivalence on the set of smooth curves in M , which pas through the point x_0 . Let $[\rho]_{x_0}$ be a class of equivalence. It will be called a "2-osculator space" in a point $x_0 \in M$. The set of 2-osculator spaces in the point $x_0 \in M$ will be denoted by $Osc_{x_0}^2 M$, and we put

$$Osc^2 M = \bigcup_{x_0 \in M} Osc_{x_0}^2 M.$$

One considers the mapping $\pi^2 : Osc^2 M \rightarrow M$ define by $\pi^2([\rho]_{x_0}) = x_0$. Obviously, π^2 is a surjection.

The set $Osc^2 M$ is endowed with a natural differentiable structure, induced by that of the manifold M , so that π^2 is a differentiable mapping. It will be descrieb bellow.

The curve $\rho : I \rightarrow M$ ($\text{Im } \rho \subset U$) is analytically represented in the local chart (U, φ) by $x_0 = x_0^a (= x^a(0))$. Taking the function f from (1.1), succesively equal to the coordinate functions x^a , then a representative of the class $[\rho]_{x_0}$ is given by

$$x^{*a}(t) = x^a(0) + t \frac{dx^a}{dt}(0) + \frac{1}{2} t^2 \frac{d^2 x^a}{dt^2}(0), \quad t \in (-\varepsilon, \varepsilon) \subset I.$$

The previous polynomials are determined by the coefficients

$$x_0^a = x^a(0), y^{(1)a} = \frac{dx^a}{dt}(0), y^{(2)a} = \frac{1}{2} \frac{d^2 x^a}{dt^2}(0). \quad (1.2)$$

Hence, the pair $\left((\pi^2)^{-1}(U), \Phi\right)$, with $\Phi([\rho]_{x_0}) = (x_0^a, y^{(1)a}, y^{(2)a}) \in R^{3n}$, $\forall [\rho]_{x_0} \in (\pi^2)^{-1}(U)$ is a local chart on $Osc^2 M$. Thus a differentiable atlas \mathcal{A}_M of the differentiable structure on the manifold M determines a differentiable atlas $\mathcal{A}_{Osc^2 M}$ on $Osc^2 M$ and therefore the triple $(Osc^2 M, \pi^2, M)$ is a differentiable bundle.

By (1.2), a transformation of local coordinates $(x^a, y^{(1)a}, y^{(2)a}) \rightarrow (\tilde{x}^a, \tilde{y}^{(1)a}, \tilde{y}^{(2)a})$ on the manifold $Osc^2 M$ is given by

$$\begin{cases} \tilde{x}^a = \tilde{x}^a(x^1, \dots, x^n), \det\left(\frac{\partial \tilde{x}^a}{\partial x^b}\right) \neq 0 \\ \tilde{y}^{(1)a} = \frac{\partial \tilde{x}^a}{\partial x^b} y^{(1)b} \\ 2\tilde{y}^{(2)a} = \frac{\partial \tilde{y}^{(1)a}}{\partial x^b} y^{(1)b} + 2 \frac{\partial \tilde{y}^{(1)a}}{\partial y^{(1)b}} y^{(2)b}. \end{cases} \quad (1.3)$$

One can see that $Osc^2 M$ is of dimension $3n$.

Let us consider the **2-tangent structure** \mathbb{J} on $Osc^2 M$

$$\mathbb{J}\left(\frac{\partial}{\partial x^a}\right) = \frac{\partial}{\partial y^{(1)a}}, \quad \mathbb{J}\left(\frac{\partial}{\partial y^{(1)a}}\right) = \frac{\partial}{\partial y^{(2)a}}, \quad \mathbb{J}\left(\frac{\partial}{\partial y^{(2)a}}\right) = 0$$

where $\left(\frac{\partial}{\partial x^a} \mid_u, \frac{\partial}{\partial y^{(1)a}} \mid_u, \frac{\partial}{\partial y^{(2)a}} \mid_u\right)$ is the natural basis of tangent space $T_u Osc^2 M$, $u \in Osc^2 M$. If N is a nonlinear connection on $Osc^2 M$, then $N_0 = N$, $\mathbb{J}(N_0) = N_1$ are two distributions geometrically defined on $Osc^2 M$, all of dimension n . Let us consider the distributions V_2 on $Osc^2 M$ locally generated by the vector fields $\left\{\frac{\partial}{\partial y^{(2)a}}\right\}$. Consequently, the tangent bundle to $Osc^2 M$ at the point $u \in Osc^2 M$ is given by a direct sum of the vector space:

$$T_u Osc^2 M = N_0(u) \oplus N_1(u) \oplus V_2(u), \forall u \in Osc^2 M. \quad (1.4)$$

We consider $\left\{\frac{\delta}{\delta x^a}, \frac{\delta}{\delta y^{(1)a}}, \frac{\delta}{\delta y^{(2)a}}\right\}$ the adapted basis to the decomposition

(1.4) and its dual basis denoted by $(dx^a, \delta y^{(1)a}, \delta y^{(2)a})$, where

$$\begin{cases} \frac{\delta}{\delta x^a} = \frac{\partial}{\partial x^a} - N_{(1)a}^b \frac{\delta}{\delta y^{(1)b}} - N_{(2)a}^b \frac{\partial}{\partial y^{(2)b}} \\ \frac{\delta}{\delta y^{(1)a}} = \frac{\partial}{\partial y^{(1)a}} - N_{(1)a}^b \frac{\partial}{\partial y^{(2)b}} \\ \frac{\delta}{\delta y^{(2)a}} = \frac{\partial}{\partial y^{(2)a}} \end{cases} \quad (1.5)$$

and

$$\begin{cases} dx^a = dx^a \\ \delta y^{(1)a} = dy^{(1)a} + M_{(1)b}^a dx^b \\ \delta y^{(2)a} = dy^{(2)a} + M_{(1)b}^a \delta y^{(1)b} + M_{(2)b}^a \delta y^{(2)b}. \end{cases} \quad (1.6)$$

Definition 1.1 A linear connection D on $Osc^2 M$ is called **N-linear connection** if it preserves by parallelism the horizontal and vertical distributions N_0, N_1 and V_2 on $Osc^2 M$.

Any N-linear connection D can be represented by an unique system of functions $D\Gamma(N) = \left(L_{(i0)bd}^a, C_{(i1)bd}^a, C_{(i2)bd}^a \right), (i = 0, 1, 2)$. These functions are called **the coefficients** of the N-linear connection D.

If on the manifold $Osc^2 M$ is given a N-linear connection D then there exists a h_i -, v_{1i} - and v_{2i} -**covariant derivatives** in local adapted basis $(i = 0, 1, 2)$.

Any d-tensor T , of type (r, s) can be represented in the adapted basis and its dual basis in the form

$$T = T_{b_1 \dots b_s}^{a_1 \dots a_r} \delta_{a_1} \otimes \dots \otimes \delta_{a_r} \otimes dx^{b_1} \otimes \dots \otimes \delta y^{(2)b_s}.$$

and we have

$$\begin{aligned} T_{b_1 \dots b_s}^{a_1 \dots a_r} \Big|_{id} &= \delta_a T_{b_1 \dots b_s}^{a_1 \dots a_r} + L_{(i0)cd}^{a_1} T_{b_1 \dots b_s}^{ca_2 \dots a_r} + \dots + \\ &+ L_{(i0)cd}^{a_r} T_{b_1 \dots b_s}^{a_1 \dots a_{r-1}c} - L_{(i0)b_1d}^c T_{cb_2 \dots b_s}^{a_1 \dots a_r} - \dots - L_{(i0)b_sd}^c T_{cb_2 \dots b_{s-1}c}^{a_1 \dots a_r}, \\ T_{b_1 \dots b_s}^{a_1 \dots a_r} \Big|_{id}^{(1)} &= \delta_{1a} T_{b_1 \dots b_s}^{a_1 \dots a_r} + C_{(i1)cd}^{a_1} T_{b_1 \dots b_s}^{ca_2 \dots a_r} + \dots + \\ &+ C_{(i1)cd}^{a_r} T_{b_1 \dots b_s}^{a_1 \dots a_{r-1}c} - C_{(i1)b_1d}^c T_{cb_2 \dots b_s}^{a_1 \dots a_r} - \dots - C_{(i1)b_sd}^c T_{cb_2 \dots b_{s-1}c}^{a_1 \dots a_r}, \end{aligned}$$

$$\begin{aligned}
T_{b_1 \dots b_s}^{a_1 \dots a_r} \Big|_{id}^{(2)} &= \delta_{2a} T_{b_1 \dots b_s}^{a_1 \dots a_r} + C_{(i2)cd}^{a_1} T_{b_1 \dots b_s}^{ca_2 \dots a_r} + \dots + \\
&+ C_{(i2)cd}^{a_r} T_{b_1 \dots b_s}^{a_1 \dots a_{r-1}c} - C_{(i2)b_1d}^c T_{cb_2 \dots b_s}^{a_1 \dots a_r} - \dots - C_{(i2)b_sd}^c T_{cb_2 \dots b_{s-1}c}^{a_1 \dots a_r}, \\
\left(\delta_{1a} &= \frac{\delta}{\delta y^{(1)a}}, \delta_{2a} = \frac{\delta}{\delta y^{(2)a}}; i = 0, 1, 2 \right).
\end{aligned}$$

The operators " $\Big|_{id}^{(1)}$ ", " $\Big|_{id}^{(2)}$ " and " $\Big|_{id}^{(2)}$ " are called the h_{i-}, v_{1i} - and v_{2i} -**covariant derivatives** with respect to $D\Gamma(N)$.

Definition 1.2 A **metric structure** on the manifold $Osc^2(M)$ is a symmetric covariant tensor field \mathbb{G} of the type $(0, 2)$ which is non degenerate at each point $u \in Osc^2(M)$ and of constant signature on $Osc^2(M)$.

Locally, a metric structure looks as follows:

$$\mathbb{G} = g_{ab} dx^a \otimes dx^b + g_{ab}^{(1)} \delta y^{(1)a} \otimes \delta y^{(1)b} + g_{ab}^{(2)} \delta y^{(2)a} \otimes \delta y^{(2)b},$$

where

$$rank \left\| g_{ab}^{(i)} \right\| = n, (i = 0, 1, 2).$$

Definition 1.3 A N-linear connection D on $Osc^2 M$ endowed with a metric structure \mathbb{G} is said to be a **metric N-linear connection** if $D_X \mathbb{G} = 0$ for every $X \in \mathcal{X}(Osc^2 M)$.

2 Submanifolds in the manifold $Osc^2 M$

Let M be a C^∞ real, n-dimensional manifold and \check{M} be a real, m-dimensional manifold, immersed in M through the immersion $i : \check{M} \rightarrow M$. Locally, i can be given in the form

$$x^a = x^a(u^1, \dots, u^m), \quad rank \left\| \frac{\partial x^a}{\partial u^\alpha} \right\| = m. \quad (2.1)$$

The indices a, b, c, \dots run over the set $\{1, \dots, n\}$ and $\alpha, \beta, \gamma, \dots$ run on the set $\{1, \dots, m\}$. We assume $1 < m < n$. If i is an embedding, then we identify \check{M} to $i(\check{M})$ and say that \check{M} is a *submanifold* of the manifold M . Therefore (2.1) will be called the *parametric equations of the submanifold \check{M}* in the manifold M .

The embedding $i : \check{M} \rightarrow M$ determines an immersion $Osc^2 i : Osc^2 \check{M} \rightarrow Osc^2 M$, defined by the covariant functor $Osc^2 : Man \rightarrow Man$ [6].

The mapping $Osc^2 i : Osc^2 \check{M} \rightarrow Osc^2 M$ has the parametric equations:

$$\begin{cases} x^a = x^a(u^1, \dots, u^m), \text{rank} \left\| \frac{\partial x^a}{\partial u^\alpha} \right\| = m \\ y^{(1)a} = \frac{\partial x^a}{\partial u^\alpha} v^{(1)\alpha} \\ 2y^{(2)a} = \frac{\partial y^{(1)a}}{\partial u^\alpha} v^{(1)\alpha} + 2 \frac{\partial y^{(1)a}}{\partial v^{(1)\alpha}} v^{(2)\alpha} \end{cases} \quad (2.2)$$

where

$$\begin{cases} \frac{\partial x^a}{\partial u^\alpha} = \frac{\partial y^{(1)a}}{\partial v^{(1)\alpha}} = \frac{\partial y^{(2)a}}{\partial v^{(2)\alpha}} \\ \frac{\partial y^{(1)a}}{\partial u^\alpha} = \frac{\partial y^{(2)a}}{\partial v^{(1)\alpha}}. \end{cases} \quad (2.3)$$

The Jacobian matrix of (2.2) is $J(Osc^2 i)$ and it has the rank equal to $3m$. So, $Osc^2 i$ is an immersion. The differential i_* of the mapping $Osc^2 i : Osc^2 \check{M} \rightarrow Osc^2 M$ leads to the relation between the natural basis of the modules $\mathcal{X}(Osc^2 \check{M})$ and $\mathcal{X}(Osc^2 M)$ given by

$$i_* \left\| \frac{\partial}{\partial u^\alpha} \frac{\partial}{\partial v^{(1)\alpha}} \dots \frac{\partial}{\partial v^{(2)\alpha}} \right\| = \left\| \frac{\partial}{\partial x^a} \frac{\partial}{\partial y^{(1)a}} \dots \frac{\partial}{\partial y^{(2)a}} \right\| J(Osc^2 i).$$

i_* maps the cotangent space $T^*(Osc^2 M)$ in a point of $Osc^2 M$, into the cotangent space $T^*(Osc^2 \check{M})$ in a point of $Osc^2 \check{M}$ by the rule:

$$\begin{aligned} dx^a &= \frac{\partial x^a}{\partial u^\alpha} du^\alpha \\ dy^{(1)a} &= \frac{\partial y^{(1)a}}{\partial u^\alpha} du^\alpha + \frac{\partial y^{(1)a}}{\partial v^{(1)\alpha}} dv^{(1)\alpha} \\ dy^{(2)a} &= \frac{\partial y^{(2)a}}{\partial u^\alpha} du^\alpha + \frac{\partial y^{(2)a}}{\partial v^{(1)\alpha}} dv^{(1)\alpha} + \frac{\partial y^{(2)a}}{\partial v^{(2)\alpha}} dv^{(2)\alpha} \end{aligned} \quad (2.4)$$

We used the previous theory to study the induced geometrical object fields from $Osc^2 M$ to $Osc^2 \check{M}$.

Let us consider a Finsler space, $F^n = (M, F(x, y^{(1)}))$ having $g_{ab} = \frac{1}{2} \frac{\partial F^2}{\partial y^{(1)a} y^{(1)b}}$ as fundamental tensor field. The restriction \check{F} of the function F to the manifold $Osc^2 \check{M}$ is given by

$$\check{F}(u, v^{(1)}) = F(x(u), y^{(1)}(u, v^{(1)}))$$

and the pair $\check{F}^n = (M, \check{F})$ is a Finsler space. \check{F}^n is called the **induced Finsler subspaces** of the Finsler space F^n .

Next, we consider

$$B_\alpha^a = \frac{\partial x^a}{\partial u^\alpha}. \quad (2.5)$$

and $\mathbb{G} = g_{ab} dx^a \otimes dx^b + g_{ab} \delta y^{(1)a} \otimes \delta y^{(1)a} + g_{ab} \delta y^{(2)a} \otimes \delta y^{(2)a}$ the Sasaki prolongation of the fundamental tensor g along $Osc^2 M$.

There exist a nonlinear connection on the manifold $Osc^2 M$ determined only by $g_{ab}(x, y^{(1)})$. The dual coefficients of this nonlinear connection are [6]:

$$\begin{aligned} M_1^a{}_b &= \frac{\partial G^a}{\partial y^{(1)b}}, \\ M_2^a{}_b &= \frac{1}{2} \left(\Gamma M_1^a{}_b + M_1^a{}_d M_1^d{}_b \right), \end{aligned} \quad (2.6)$$

where

$$\begin{aligned} G^a &= \frac{1}{2} \gamma_{bc}^a(x, y) y^{(1)b} y^{(1)c}, \\ \Gamma &= y^{(1)a} \frac{\partial}{\partial x^a} + 2y^{(2)a} \frac{\partial}{\partial y^{(1)a}}, \end{aligned}$$

and $\gamma_{bc}^a(x, y^{(1)})$ are the Christoffel symbols of the fundamental tensor g ,

$$\gamma_{bc}^a(x, y^{(1)}) = \frac{1}{2} g^{ad} \left(\frac{\partial g_{dc}}{\partial x^b} + \frac{\partial g_{bd}}{\partial x^c} - \frac{\partial g_{bc}}{\partial x^d} \right).$$

Thus, $\{B_1^a, B_2^a, \dots, B_m^a\}$ are m-linear independent d-vector fields on $Osc^2 \check{M}$. Also, $\{B_\alpha^1, B_\alpha^2, \dots, B_\alpha^n\}$ are d-covector fields, with respect to the next transformations of coordinates:

$$\left\{ \begin{aligned} \bar{u}^\alpha &= \bar{u}^\alpha(u^1, \dots, u^m), \text{rank} \left\| \frac{\partial \bar{u}^\alpha}{\partial u^\beta} \right\| = m \\ \bar{v}^{(1)\alpha} &= \frac{\partial \bar{u}^\alpha}{\partial u^\beta} v^{(1)\beta} \\ 2\bar{v}^{(2)\alpha} &= \frac{\partial \bar{v}^{(1)\alpha}}{\partial u^\beta} v^{(1)\beta} + 2 \frac{\partial \bar{v}^{(1)\alpha}}{\partial v^{(1)\beta}} v^{(2)\beta}. \end{aligned} \right. \quad (2.7)$$

Of course, d-vector fields $\{B_1^a, \dots, B_m^a\}$ are tangent to the submanifold \check{M} .

We say that a d-vector field $\xi^a(x, y^{(1)}, y^{(2)})$ is **normal** to $Osc^2 \check{M}$ if, on $\check{\pi}^{-1}(\check{U}) \subset Osc^2 \check{M}$, we have

$$g_{ab}(x(u), y^{(1)}(u, v^{(1)})) B_\alpha^a(u) \cdot \xi^b(x(u), y^{(1)}(u, v^{(1)}), y^{(2)}(u, v^{(1)}, v^{(2)})) = 0.$$

Consequently, on $\tilde{\pi}^{-1}(\tilde{U}) \subset Osc^2\tilde{M}$ there exist $n - m$ unit vector fields $B_{\bar{\alpha}}^a$, ($\bar{\alpha} = 1, \dots, n - m$) normal along $Osc^2\tilde{M}$, and to each other:

$$g_{ab}B_{\bar{\alpha}}^aB_{\bar{\beta}}^b = 0, \quad g_{ab}B_{\bar{\alpha}}^aB_{\bar{\beta}}^b = \delta_{\bar{\alpha}\bar{\beta}}, \quad (\bar{\alpha}, \bar{\beta} = 1, \dots, n - m). \quad (2.8)$$

The system of d-vectors $B_{\bar{\alpha}}^a$ ($\bar{\alpha} = 1, \dots, n - m$) is determined up to orthogonal transformations of the form

$$B_{\bar{\alpha}'}^a = A_{\bar{\alpha}'}^{\bar{\beta}} B_{\bar{\beta}}^a, \quad \|A_{\bar{\alpha}'}^{\bar{\beta}}\| \in \mathcal{O}(n - m), \quad (2.9)$$

where $\bar{\alpha}, \bar{\beta}, \dots$ run over the set $\{1, 2, \dots, n - m\}$.

We say that the system of d-vectors $\{B_{\bar{\alpha}}^a, B_{\bar{\alpha}}^a\}$ determines a frame in Osc^2M along to $Osc^2\tilde{M}$.

Its dual frame will be denoted by $\{B_a^\alpha(u, v^{(1)}, v^{(2)}), B_{\bar{\alpha}}^a(u, v^{(1)}, v^{(2)})\}$. This is also defined on an open set $\tilde{\pi}^{-1}(\tilde{U}) \subset Osc^2\tilde{M}$, \tilde{U} being a domain of a local chart on the submanifold \tilde{M} .

The conditions of duality are given by:

$$B_{\bar{\beta}}^a B_a^\alpha = \delta_{\bar{\beta}}^\alpha, \quad B_{\bar{\beta}}^a B_a^{\bar{\alpha}} = 0, \quad B_a^\alpha B_{\bar{\beta}}^a = 0, \quad B_a^{\bar{\alpha}} B_{\bar{\beta}}^a = \delta_{\bar{\beta}}^{\bar{\alpha}} \quad (2.10)$$

$$B_{\bar{\alpha}}^a B_b^\alpha + B_{\bar{\alpha}}^a B_b^{\bar{\alpha}} = \delta_b^a. \quad (2.11)$$

Using (2.8), we deduce:

$$g_{\alpha\beta}B_a^\alpha = g_{ab}B_{\bar{\beta}}^a, \quad \delta_{\bar{\alpha}\bar{\beta}}B_b^{\bar{\beta}} = g_{ab}B_{\bar{\alpha}}^a. \quad (2.12)$$

So, we can look to the set

$$\mathcal{R} = \left\{ \left(u, v^{(1)}, v^{(2)} \right); B_a^\alpha(u), B_{\bar{\alpha}}^a(u, v^{(1)}, v^{(2)}) \right\}$$

$(u, v^{(1)}, v^{(2)}) \in \tilde{\pi}^{-1}(\tilde{U})$ as a moving frame. Now, we shall represent in \mathcal{R} the d-tensor fields from the space Osc^2M , restricted to the open set $\tilde{\pi}^{-1}(\tilde{U})$.

3 Induced nonlinear connections

Now, let us consider the nonlinear connection N (2.6) on the manifold Osc^2M . Then its dual coefficients $M_{(1)b}^a, M_{(2)b}^a$ depends only by the metric g . We will prove that the restriction of the of the nonlinear connection N to $Osc^2\tilde{M}$ uniquely determines an induced nonlinear connection \tilde{N} on $Osc^2\tilde{M}$. Of course, \tilde{N} is well determined by means of its dual coefficients $\left(\tilde{M}_{(1)\bar{\beta}}^\alpha, \tilde{M}_{(2)\bar{\beta}}^\alpha \right)$ or by means of its adapted cobasis $(du^\alpha, \delta v^{(1)\alpha}, \delta v^{(2)\alpha})$.

Definition 3.1 A non-linear connection \tilde{N} on $Osc^2\tilde{M}$ is called **induced** by the nonlinear connection N if we have

$$\delta v^{(1)\alpha} = B_a^\alpha \delta y^{(1)a}, \quad \delta v^{(2)\alpha} = B_a^\alpha \delta y^{(2)a} \quad (3.1)$$

Proposition 3.1 *The dual coefficients of the nonlinear connection \check{N} are*

$$\begin{aligned}\check{M}_{(1)\beta}^\alpha &= B_a^\alpha \left(B_{0\beta}^a + M_{(1)b}^a B_\beta^b \right) \\ \check{M}_{(2)\beta}^\alpha &= B_a^\alpha \left(\frac{1}{2} \frac{\partial B_{\delta\gamma}^a}{\partial u^\beta} v^{(1)\delta} v^{(1)\gamma} + B_{\delta\beta}^a v^{(2)\delta} + M_{(1)b}^a B_{0\beta}^b + M_{(2)b}^a B_\beta^b \right)\end{aligned}\quad (3.2)$$

where $M_{(1)b}^a, M_{(2)b}^a$ are the dual coefficients of the non-linear connection N .

Theorem 3.1 *The cobasis $(dx^a, \delta y^{(1)a}, \delta y^{(2)a})$ restricted to $Osc^2 \check{M}$ is uniquely represented in the moving frame \mathcal{R} in the following form:*

$$\begin{cases} dx^a = B_\beta^a du^\beta \\ \delta y^{(1)a} = B_\alpha^a \delta v^{(1)\alpha} + B_{\bar{\alpha}(1)\beta}^a K_{\bar{\beta}}^{\bar{\alpha}} du^\beta \\ \delta y^{(2)a} = B_\alpha^a \delta v^{(2)\alpha} + B_{\bar{\beta}(1)\alpha}^a K_{\bar{\alpha}}^{\bar{\beta}} \delta v^{(1)\alpha} + B_{\bar{\beta}(2)\alpha}^a K_{\bar{\alpha}}^{\bar{\beta}} du^\alpha \end{cases} \quad (3.3)$$

where

$$\begin{aligned}K_{(1)\beta}^{\bar{\alpha}} &= B_a^{\bar{\alpha}} \left(B_{0\beta}^a + M_{(1)b}^a B_\beta^b \right) \\ K_{(2)\beta}^{\bar{\alpha}} &= B_a^{\bar{\alpha}} \left(\frac{1}{2} \frac{\partial B_{\delta\gamma}^a}{\partial u^\beta} v^{(1)\delta} v^{(1)\gamma} + B_{\delta\beta}^a v^{(2)\delta} + M_{(1)b}^a B_{0\beta}^b + M_{(2)b}^a B_\beta^b - \right. \\ &\quad \left. - B_f^{\bar{\alpha}} B_d^\gamma \left(B_\gamma^f + M_{(1)b}^f B_\gamma^b \right) \left(B_{0\beta}^d + M_{(1)g}^d B_\beta^g \right) \right)\end{aligned}\quad (3.4)$$

are mixed d-tensor fields.

Proof. *The first relation is obviously. From 2.2 and 3.2 we obtain 3.3. ■*

Generally, a set of functions $T_{j\dots\beta\dots\bar{\beta}}^{i\dots\alpha\dots\bar{\alpha}}(u, v^{(1)}, v^{(2)})$ which are d-tensors in the index a, b, \dots , and d-tensors in the index α, β, \dots and tensors with respect to the transformations (2.9) in the index $\bar{\alpha}, \bar{\beta}, \dots$ is called a *mixed* d-tensor field on the manifold $Osc^2 \check{M}$.

4 The relative covariant derivatives

We shall construct the operators $\nabla_{(i)}$ of relative (or mixed) covariant derivation in the algebra of mixed d-tensor fields. It is clear that $\nabla_{(i)}$ will be known if its action of functions and on the vector fields of the form

$$\begin{aligned}X^a(x(u), y^{(1)}(u, v^{(1)}), y^{(2)}(u, v^{(1)}, v^{(2)})) \\ X^\alpha(u, v^{(1)}, v^{(2)}), X^{\bar{\alpha}}(u, v^{(1)}, v^{(2)})\end{aligned}\quad (4.1)$$

are known.

Let D the canonical metrical N -linear connection on the manifold $Osc^2 M$ ([1],[3],[4])

$$\begin{aligned}
\overset{c}{L}_{(00)}^a{}_{bc} &= \frac{1}{2}g^{ad} \left(\delta_b g_{dc} + \delta_c g_{bd} - \delta_d g_{bc} \right), \\
\overset{c}{L}_{(j0)}^a{}_{bc} &= \overset{B}{(jj)}_c{}^a + \frac{1}{2}g^{ad} \left(\delta_c g_{bd} - \overset{f}{B}_{(jj)}^f{}_{cb} g_{fd} - \overset{f}{B}_{(jj)}^f{}_{cd} g_{bf} \right), (j = 1, 2) \\
\overset{c}{C}_{(k1)}^a{}_{bc} &= \frac{1}{2}g^{ad} \delta_{1c} g_{bd}, (k = 0, 2) \\
\overset{c}{C}_{(l2)}^a{}_{bc} &= \frac{1}{2}g^{ad} \delta_{2c} g_{bd}, (l = 0, 1), \\
\overset{c}{C}_{(ii)}^a{}_{bc} &= \frac{1}{2}g^{ad} \left(\delta_{ib} g_{dc} + \delta_{ic} g_{bd} - \delta_{id} g_{bc} \right), (i = 1, 2).
\end{aligned} \tag{4.2}$$

Definition 4.1 The **coupling** of the canonical metrical N -linear connection D to the induced nonlinear connection \check{N} along $Osc^2 \check{M}$ is locally given by the set of its nine coefficients $\check{D}\Gamma(\check{N}) = \left(\check{\overset{a}{L}}_{(i0)}^a{}_{b\delta}, \check{\overset{a}{C}}_{(i1)}^a{}_{b\delta}, \check{\overset{a}{C}}_{(i2)}^a{}_{b\delta} \right), (i = 0, 1, 2)$, where

$$\begin{aligned}
\check{\overset{a}{L}}_{(i0)}^a{}_{b\delta} &= \overset{L}{(i0)}_b{}^a B_\delta^d + \overset{C}{(i1)}_b{}^a B_\delta^d K_{(1)}^{\bar{\delta}} + \overset{C}{(i2)}_b{}^a B_\delta^d K_{(2)}^{\bar{\delta}} \\
\check{\overset{a}{C}}_{(i1)}^a{}_{b\delta} &= \overset{C}{(i1)}_b{}^a B_\delta^d + \overset{C}{(i2)}_b{}^a B_\delta^d K_{(1)}^{\bar{\delta}} \quad (i = 0, 1, 2) \\
\check{\overset{a}{C}}_{(i2)}^a{}_{b\delta} &= \overset{C}{(i2)}_b{}^a B_\delta^d.
\end{aligned} \tag{4.3}$$

We have the operators $\check{D}_{(i)}$ and $D_{(i)}$ ($i = 0, 1, 2$) with the property

$$\check{D}_{(i)} X^a = D_{(i)} X^a \text{ (modulo 3.3)}, \tag{4.4}$$

where

$$D_{(i)} X^a = dX^a + X^b \omega_{(i)b}^a, \tag{4.5}$$

and

$$\check{D}_{(i)} X^a = dX^a + X^b \check{\omega}_{(i)b}^a. \tag{4.6}$$

$\omega_{(i)b}^a$ are the 1-forms of the canonical metrical N -connection D . $\check{\omega}_{(i)b}^a$ are **coupling 1-forms** of the coupling \check{D} respectively.

Of course, we can write $\check{D}_{(i)} X^a$ in the form

$$\check{D}_{(i)} X^a = X^a|_{i\delta} du^\delta + X^a|_{i\delta}^{(1)} \delta v^{(1)\delta} + X^a|_{i\delta}^{(2)} \delta v^{(2)\delta}.$$

Definition 4.2 We call the **induced tangent N -linear connection** on $Osc^2 \check{M}$ by the canonical metrical N -linear connection D the set of its nine coefficients $D^\top \Gamma(\check{N}) = \left(L_{(i0)\beta\delta}^\alpha, C_{(i1)\beta\delta}^\alpha, C_{(i2)\beta\delta}^\alpha \right)$ ($i = 0, 1, 2$), where

$$\begin{aligned} L_{(i0)\beta\delta}^\alpha &= B_d^\alpha \left(B_{\beta\delta}^d + B_\beta^f \check{L}_{(i0)f\delta}^d \right) \\ C_{(i1)\beta\delta}^\alpha &= B_d^\alpha B_\beta^f \check{C}_{(i1)f\delta}^d \quad (i = 0, 1, 2) \\ C_{(i2)\beta\delta}^\alpha &= B_d^\alpha B_\beta^f \check{C}_{(i2)f\delta}^d. \end{aligned} \quad (4.7)$$

We have the operators $D_{(i)}^\top$ with the properties

$$D_{(i)}^\top X^\alpha = B_b^\alpha \check{D}_{(i)} X^b, \quad \text{for } X^a = B_\gamma^a X^\gamma \quad (4.8)$$

$$D_{(i)}^\top X^\alpha = dX^\alpha + X^\beta \omega_{(i)\beta}^\alpha, \quad (4.9)$$

where $\omega_{(i)\beta}^\alpha$ are the **induced tangent connection 1-forms** of $D_{(i)}^\top$ ($i = 0, 1, 2$).

As in the case of \check{D} we may write

$$D_{(i)}^\top X^\alpha = X|_{i\delta}^\alpha du^\delta + X^\alpha|_{i\delta}^{(1)} \delta v^{(1)\delta} + X^\alpha|_{i\delta}^{(2)} \delta v^{(2)\delta}.$$

Definition 4.3 We call the **induced normal N -linear connection** on $Osc^2 \check{M}$ by the canonical metrical N -linear connection D the set of its nine coefficients $D^\perp \Gamma(\check{N}) = \left(L_{(i0)\bar{\beta}\delta}^{\bar{\alpha}}, C_{(i1)\bar{\beta}\delta}^{\bar{\alpha}}, C_{(i2)\bar{\beta}\delta}^{\bar{\alpha}} \right)$, where

$$\begin{aligned} L_{(i0)\bar{\beta}\delta}^{\bar{\alpha}} &= B_d^{\bar{\alpha}} \left(\frac{\delta B_{\bar{\beta}}^d}{\delta u^\delta} + B_{\bar{\beta}}^f \check{L}_{(i0)f\delta}^d \right) \\ C_{(i1)\bar{\beta}\delta}^{\bar{\alpha}} &= B_d^{\bar{\alpha}} \left(\frac{\delta B_{\bar{\beta}}^d}{\delta v^{(1)\delta}} + B_{\bar{\beta}}^f \check{C}_{(i1)f\delta}^d \right) \quad (i = 0, 1, 2) \\ C_{(i2)\bar{\beta}\delta}^{\bar{\alpha}} &= B_d^{\bar{\alpha}} \left(\frac{\partial B_{\bar{\beta}}^d}{\partial v^{(2)\delta}} + B_{\bar{\beta}}^f \check{C}_{(i2)f\delta}^d \right). \end{aligned} \quad (4.10)$$

As before, we have the operators $D_{(i)}^\perp$ with the properties

$$D_{(i)}^\perp X^{\bar{\alpha}} = B_b^{\bar{\alpha}} \check{D}_{(i)} X^b, \quad \text{for } X^a = B_{\bar{\gamma}}^a X^{\bar{\gamma}} \quad (4.11)$$

$$D_{(i)}^\perp X^{\bar{\alpha}} = dX^{\bar{\alpha}} + X^{\bar{\beta}} \omega_{(i)\bar{\beta}}^{\bar{\alpha}} \quad (4.12)$$

where $\omega_{(i)\bar{\beta}}^{\bar{\alpha}}$ are the **induced normal connection 1-forms** of $D_{(i)}^\perp$ ($i = 0, 1, 2$).

We may set

$$D_{(i)}^\perp X^{\bar{\alpha}} = X_{|i\delta}^{\bar{\alpha}} du^\delta + X^{\bar{\alpha}} \big|_{i\delta}^{(1)} \delta v^{(1)\delta} + X^{\bar{\alpha}} \big|_{i\delta}^{(2)} \delta v^{(2)\delta}.$$

Now, we can define the relative (or mixed) covariant derivatives $\nabla_{(i)}$ enounced at the begining of this section.

Theorem 4.4 *A relative (mixed) covariant derivation in the algebra of mixed d-tensor fields is an operator $\nabla_{(i)}$ for which the following properties hold:*

$$\nabla_{(i)} f = df, \quad \forall f \in \mathcal{F}(Osc^2 \check{M})$$

$$\nabla_{(i)} X^a = \check{D}_{(i)} X^a, \quad \nabla_{(i)} X^\alpha = D_{(i)}^\top X^\alpha, \quad \nabla_{(i)} X^{\bar{\alpha}} = D_{(i)}^\perp X^{\bar{\alpha}} \quad (i = 0, 1, 2)$$

The connection 1-forms $\check{\omega}_{(i)b}^a, \omega_{(i)\beta}^\alpha, \omega_{(i)\bar{\beta}}^{\bar{\alpha}}$ will be called the **connection 1-forms** of $\nabla_{(i)}$.

The **Liouville vector fields** of the submanifold $Osc^2 \check{M}$, introduce by R. Miron in [6], are

$$\begin{aligned} \mathbb{C}_1 &= v^{(1)\alpha} \frac{\partial}{\partial v^{(2)\alpha}} \\ \mathbb{C}_2 &= v^{(1)\alpha} \frac{\partial}{\partial v^{(1)\alpha}} + 2v^{(2)\alpha} \frac{\partial}{\partial v^{(2)\alpha}}. \end{aligned}$$

If we represent this vector fields in the adapted basis, we get

$$\mathbb{C}_1 = z^{(1)\alpha} \dot{\partial}_{2\alpha}, \quad \mathbb{C}_2 = z^{(1)\alpha} \delta_{1\alpha} + 2z^{(2)\alpha} \dot{\partial}_{2\alpha}$$

where

$$z^{(1)\alpha} = v^{(1)\alpha}, \quad z^{(2)\alpha} = v^{(2)\alpha} + \frac{1}{2} M_{(1)}^\alpha{}_\beta v^{(1)\beta}.$$

The d -vector fields $z^{(1)\alpha}$ and $z^{(2)\alpha}$ are called the **Liouville d-vector fields** of the submanifold $Osc^2 \check{M}$.

The $(z^{(1)})$ - and $(z^{(2)})$ -**deflection** tensor fields are:

$$\begin{aligned} z^{(1)\alpha}{}_{|i\beta} &= D_i^\alpha{}_\beta, & z^{(1)\alpha}{}_{|i\beta} &= d_i^{(11)}{}^\alpha{}_\beta, & z^{(1)\alpha}{}_{|i\beta} &= d_i^{(12)}{}^\alpha{}_\beta, \\ z^{(2)\alpha}{}_{|i\beta} &= D_i^\alpha{}_\beta, & z^{(2)\alpha}{}_{|i\beta} &= d_i^{(21)}{}^\alpha{}_\beta, & z^{(2)\alpha}{}_{|i\beta} &= d_i^{(22)}{}^\alpha{}_\beta. \end{aligned} \quad (4.13)$$

Proposition 4.1 The $(z^{(1)})$ -deflection tensor fields have the expressions:

$$\begin{aligned} D_i^\alpha{}_\beta &= -N_1^\alpha{}_\beta + z^{(1)\gamma} L_{(i0)}^\alpha{}_{\gamma\beta}, \\ d_i^{(11)}{}^\alpha{}_\beta &= \delta_\beta^\alpha + z^{(1)\gamma} C_{(i1)}^\alpha{}_{\gamma\beta}, \\ d_i^{(12)}{}^\alpha{}_\beta &= z^{(1)\gamma} C_{(i2)}^\alpha{}_{\gamma\beta}, \quad (i = 0, 1, 2). \end{aligned} \quad (4.14)$$

Indeed, if we take

$$\begin{aligned} z^{(1)\alpha}{}_{|i\beta} &= \delta_\beta^\alpha z^{(1)\alpha} + z^{(1)\gamma} L_{(i0)}^\alpha{}_{\gamma\beta}, \\ z^{(1)\alpha}{}_{|i\beta} &= \delta_{j\beta} z^{(1)\alpha} + z^{(1)\gamma} C_{(ij)}^\alpha{}_{\gamma\beta}, \quad (i = 0, 1, 2; j = 1, 2; \delta_{2\beta} = \dot{\delta}_{2\beta}) \end{aligned}$$

we find this formulae.

Proposition 4.2 The $(z^{(2)})$ -deflection tensor fields are given by

$$\begin{aligned} D_i^\alpha{}_\beta &= \frac{1}{2} \left(N_2^\alpha{}_\beta + M_2^\alpha{}_\beta \right) + \frac{1}{2} z^{(1)\gamma} \delta_\beta N_1^\alpha{}_\gamma + z^{(2)\gamma} L_{(i0)}^\alpha{}_{\gamma\beta}, \\ d_i^{(21)}{}^\alpha{}_\beta &= -\frac{1}{2} \left(2N_2^\alpha{}_\beta - N_1^\alpha{}_\beta \right) + \frac{1}{2} z^{(1)\gamma} B_{11}^\alpha{}_{\gamma\beta} + z^{(2)\gamma} C_{(i1)}^\alpha{}_{\gamma\beta}, \\ d_i^{(22)}{}^\alpha{}_\beta &= \delta_\beta^\alpha + \frac{1}{2} z^{(1)\gamma} B_{12}^\alpha{}_{\gamma\beta} + z^{(2)\gamma} C_{(i2)}^\alpha{}_{\gamma\beta}, \quad (i = 0, 1, 2). \end{aligned} \quad (4.15)$$

5 The Ricci identities

Let $\check{D}\Gamma(\check{N}) = \left(\check{L}_{(i0)}^a{}_{b\delta}, \check{C}_{(i1)}^a{}_{b\delta}, \check{C}_{(i2)}^a{}_{b\delta} \right)$ the coupling of the canonical metrical N -linear connection D (4.2) to the induced nonlinear connection \check{N} along to the

manifold $Osc^2 \check{M}$, $D^\top \Gamma(\check{N}) = \left(L_{(i0)}^\alpha{}_{\beta\delta}, C_{(i1)}^\alpha{}_{\beta\delta}, C_{(i2)}^\alpha{}_{\beta\delta} \right)$ and $D^\perp \Gamma(\check{N}) = \left(L_{(i0)}^{\bar{\alpha}}{}_{\beta\delta}, C_{(i1)}^{\bar{\alpha}}{}_{\beta\delta}, C_{(i2)}^{\bar{\alpha}}{}_{\beta\delta} \right)$ ($i = 0, 1, 2$) the induced tangent N -linear connection and the induced normal N -linear connection on $Osc^2 \check{M}$, respectively.

Teoremă 5.1 For any d-vector fields X^α , the following Ricci identities hold:

$$\begin{aligned}
X^\alpha|_{i\beta|i\gamma} - X^\alpha|_{i\beta|i\gamma} &= X^\delta R_{(0i)}^\alpha{}_{\beta\gamma} - \overset{(i)}{T}_{(0)}^\sigma{}_{\beta\gamma} X^\alpha|_{i\sigma} - R_{(01)}^\sigma{}_{\beta\gamma} X^\alpha|_{i\sigma} - \\
&\quad - R_{(02)}^\sigma{}_{\beta\gamma} X^\alpha|_{i\sigma}, \\
X^\alpha|_{i\beta}|_{i\gamma} - X^\alpha|_{i\gamma}|_{i\beta} &= X^\delta P_{(1i)}^\alpha{}_{\beta\gamma} - C_{(i1)}^\sigma{}_{\beta\gamma} X^\alpha|_{i\sigma} - \overset{(i)}{P}_{(11)}^\sigma{}_{\beta\gamma} X^\alpha|_{i\sigma} \\
&\quad - P_{(12)}^\sigma{}_{\beta\gamma} X^\alpha|_{i\sigma}, \\
X^\alpha|_{i\beta}|_{i\gamma} - X^\alpha|_{i\gamma}|_{i\beta} &= X^\delta P_{(2i)}^\alpha{}_{\beta\gamma} - C_{(i2)}^\sigma{}_{\beta\gamma} X^\alpha|_{i\sigma} - \overset{(i)}{P}_{(21)}^\sigma{}_{\beta\gamma} X^\alpha|_{i\sigma} \\
&\quad - \overset{(i)}{P}_{(22)}^\sigma{}_{\beta\gamma} X^\alpha|_{i\sigma},
\end{aligned} \tag{5.1}$$

$$\begin{aligned}
X^\alpha|_{i\beta}|_{i\gamma} - X^\alpha|_{i\gamma}|_{i\beta} &= X^\delta Q_{(21)}^\alpha{}_{\beta\gamma} - C_{(i2)}^\sigma{}_{\beta\gamma} X^\alpha|_{i\sigma} - \\
&\quad - \overset{(i)}{Q}_{(22)}^\sigma{}_{\beta\gamma} X^\alpha|_{i\sigma},
\end{aligned}$$

$$\begin{aligned}
X^\alpha|_{i\beta}|_{i\gamma} - X^\alpha|_{i\gamma}|_{i\beta} &= X^\delta S_{(ji)}^\alpha{}_{\beta\gamma} - \overset{(i)}{S}_{(j)}^\sigma{}_{\beta\gamma} X^\alpha|_{i\sigma} - \\
&\quad - R_{(j2)}^\sigma{}_{\beta\gamma} X^\alpha|_{i\sigma}
\end{aligned}$$

where $R_{(22)}^\alpha{}_{\beta\gamma} = 0, (i = 0, 1, 2, j = 1, 2)$.

The Ricci identities (5.1) applied to the Liouville d-vector fields $z^{(1)\alpha}$ and $z^{(2)\alpha}$ lead to the some fundamental identities.

Theorem 5.2 *The deflection tensor fields satisfy the following identities:*

$$\begin{aligned}
D_i^{(j)\alpha} \beta|_{i\gamma} - D_i^{(j)\alpha} \gamma|_{i\beta} &= z^{(j)\delta} R_{(0i)\delta}^{\alpha} \beta\gamma - D_i^{(j)\alpha} \delta_{(0)}^{\delta} T_{(0)\beta\gamma}^{\delta} - \\
&\quad - d_i^{(j1)\alpha} \delta_{(01)}^{\delta} R_{\beta\gamma}^{\delta} - d_i^{(j2)\alpha} \delta_{(02)}^{\delta} R_{\beta\gamma}^{\delta}, \\
D_i^{(j)\alpha} \beta|_{i\gamma} - d_i^{(j1)\alpha} \gamma|_{i\beta} &= z^{(j)\delta} P_{(1i)\delta}^{\alpha} \beta\gamma - D_i^{(j)\alpha} \delta_{(i1)}^{\delta} C_{\beta\gamma}^{\delta} - \\
&\quad - d_i^{(j1)\alpha} \delta_{(11)}^{\delta} P_{\beta\gamma}^{\delta} - d_i^{(j2)\alpha} \delta_{(12)}^{\delta} P_{\beta\gamma}^{\delta}, \\
D_i^{(j)\alpha} \beta|_{i\gamma} - d_i^{(j2)\alpha} \gamma|_{i\beta} &= z^{(j)\delta} P_{(2i)\delta}^{\alpha} \beta\gamma - D_i^{(j)\alpha} \delta_{(i2)}^{\delta} C_{\beta\gamma}^{\delta} - \\
&\quad - d_i^{(j1)\alpha} \delta_{(21)}^{\delta} P_{\beta\gamma}^{\delta} - d_i^{(j2)\alpha} \delta_{(22)}^{\delta} P_{\beta\gamma}^{\delta}, \\
d_i^{(j1)\alpha} \beta|_{i\gamma} - d_i^{(j2)\alpha} \gamma|_{i\beta} &= z^{(j)\delta} Q_{(2i)\delta}^{\alpha} \beta\gamma - X^{\alpha} |_{i\sigma}^{(1)} - \\
&\quad - d_i^{(j1)\alpha} \delta_{(i2)}^{\delta} C_{\beta\gamma}^{\delta} - d_i^{(j2)\alpha} \delta_{(22)}^{\delta} Q_{\beta\gamma}^{\delta}, \\
d_i^{(ji)\alpha} \beta|_{i\gamma} - d_i^{(ji)\alpha} \gamma|_{i\beta} &= z^{(j)\delta} S_{(li)\delta}^{\alpha} \beta\gamma - \\
&\quad - d_i^{(jl)\alpha} \delta_{(i)}^{\delta} S_{\beta\gamma}^{\delta} - d_i^{(j2)\alpha} \delta_{(l2)}^{\delta} R_{\beta\gamma}^{\delta},
\end{aligned} \tag{5.2}$$

$$\left(i = 0, 1, 2; j, l = 1, 2; R_{(22)\beta\gamma}^{\alpha} = 0. \right)$$

Also, if the $(z^{(1)})$ -and $(z^{(2)})$ -deflection tensors have the following particular form

$$\begin{aligned}
D_i^{(1)\alpha} \beta &= 0, & d_i^{(11)\alpha} \beta &= \delta_{\beta}^{\alpha}, & d_i^{(12)\alpha} \beta &= 0 \\
D_i^{(2)\alpha} \beta &= 0, & d_i^{(21)\alpha} \beta &= 0, & d_i^{(22)\alpha} \beta &= \delta_{\beta}^{\alpha}
\end{aligned} \tag{5.3}$$

then, the fundamental identities from (5.2) are very important, especially for applications.

Proposition 5.2 If the deflection tensors are given by (5.3), then the following identities hold:

$$\begin{aligned}
z^{(j)\delta} R_{(0i)\delta}^{\alpha}{}_{\beta\gamma} &= R_{(0j)\delta}^{\alpha}{}_{\beta\gamma}, & z^{(1)\delta} P_{(2i)\delta}^{\alpha}{}_{\beta\gamma} &= P_{(21)\delta}^{\alpha}{}_{\beta\gamma}, & z^{(2)\delta} P_{(1i)\delta}^{\alpha}{}_{\beta\gamma} &= P_{(12)\delta}^{\alpha}{}_{\beta\gamma}, \\
z^{(j)\delta} P_{(ji)\delta}^{\alpha}{}_{\beta\gamma} &= P_{(jj)\delta}^{\alpha}{}_{\beta\gamma}, & z^{(1)\delta} Q_{(2i)\delta}^{\alpha}{}_{\beta\gamma} &= C_{(i2)\delta}^{\alpha}{}_{\beta\gamma}, & z^{(2)\delta} Q_{(2i)\delta}^{\alpha}{}_{\beta\gamma} &= \dot{Q}_{(22)\delta}^{\alpha}{}_{\beta\gamma}, \\
z^{(j)\delta} S_{(ji)\delta}^{\alpha}{}_{\beta\gamma} &= \overset{j}{S}_{(i)\delta}^{\alpha}{}_{\beta\gamma}, & z^{(1)\delta} S_{(2i)\delta}^{\alpha}{}_{\beta\gamma} &= 0, & z^{(2)\delta} S_{(1i)\delta}^{\alpha}{}_{\beta\gamma} &= R_{(12)\delta}^{\alpha}{}_{\beta\gamma}.
\end{aligned} \tag{5.4}$$

($i = 0, 1, 2; j = 1, 2$)

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